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Jerk by group theoretical methods

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Abstract. We study the second central extension of the one spatial dimensional Galilei group G. We find that the kinematic quantity '*jerk*' is connected with the physics of one of the co-adjoint orbits of the first central extension of G, as much as acceleration is associated with the physics of one of the co-adjoint orbits of G.

1. Introduction

Recently we introduced group theoretically the Galilei–Newton laws of motion with constant acceleration [1]. We have also studied the symplectic actions of the Galilei group on the dual Lie algebra of its first central extension.

Also, Schot [2] and Luzader [3] introduced a physical quantity that they separately called jerk and that Sandin recently discussed as a kinematic quantity [4]: 'The velocity of a particle is the first derivative with respect to time of its position, the acceleration is the second drivative, and the jerk is the third derivative'. Sandin established the kinematic equation for a system with constant jerk and discussed the Newton second law for that system.

Let us also recall that all the generic Hamiltonian spaces for a Lie group are orbits of the co-adjoint representation of the group on his central extended dual Lie algebra, that the analogies of the projective unitary (ray) representations quantum mechanics are, in classical mechanics, the Hamiltonian actions, and that, finally, the ray representations of the 3-spatial Galilei group have previously been studied by Inönü and Wigner [5], Levy Leblond [6] and Voisin [7].

In this article we study the connected second central extension of the Galilei Lie group. We show that the equations found by Sandin can be introduced group theoretically in the same way as we have introduced Galilei–Newton laws in [1]. We also show that position can be seen as the time rate of change of some observable canonically conjugated with force.

This article is organized as follows. In section 2, we review some notions on central extensions of Lie groups and Lie algebras. In section 3, for the sake of completeness, we remind the reader the essentials of the paper [1]. The connected second central extension of the Galilei group G is introduced in section 4, where we study the physics of one of the orbits of the first central extension of G. Mass and jerk appear as characteristics of that orbit.

In the appendix, we show how the symplectic geometry is introduced on the co-adjoint orbits of the Lie group.

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2. Central extensions of Lie groups and Lie algebras [8–10]

Let G be a Lie group. A 2-cocycle c is a real function

$$c : GxG \longrightarrow \mathcal{R}^n$$

such that

$$c(g,h) + c(gh,k) = c(g,hk) + c(h,k) \tag{2.1a}$$

$$c(e,e) = 0 \tag{2.1b}$$

where e is the unit element of G. A 2-cocycle c is trivial if there exists a coboundary $b : G \to \mathcal{R}^n$ with $b(e) = 0$ and such that

$$c(g,h) = b(gh) - b(g) - b(h).$$

Two 2-cocyles c_1 and c_2 are said to be equivalent if there exists a trivial 2-cocycle c_3 such that $c_1 = c_2 + c_3$. The equivalence classes form the second cohomological group $H^2(G, \mathcal{R}^n)$ whose dimension is n.

Let G be a Lie group and let c be a 2-cocycle of G. One can then define a Lie group $G_c = \{(\theta, g) : \theta \in \mathcal{R}^n, g \in G\}$ with multiplication law:

$$(\theta, g)(\theta', g') = (\theta + \theta' + c(g, g'), gg'). \tag{2.3}$$

Associativity is ensured by (2.1a). The subgroup $\Theta = \{(\theta, e) : \theta \in \mathcal{R}^n\}$ is isomorphic to $\mathcal{R}^n$ and is central invariant in G_c. On the other hand, the quotient group G_c/Θ is isomorphic to G and G_c is said to be a central extension of G by $\mathcal{R}^n$ via the 2-cocycle c. Two equivalent 2-cocycles give rise to isomorphic central extensions. Moreover a trivial 2-cocycle gives rise to a trivial extension, a direct product of $\mathcal{R}^n$ and G.

Similarly, given a 2-cocycle $c^{(1)}$ of G_c, we can define a central extension of G_c which we denote by $G^{(2)}_{c^{(1)}}$. G_c and $G^{(2)}_{c^{(1)}}$ are, respectively, called a first central extension and a second central extension of G. A nth central extension of G is similarly constructed and is denoted $G^{(n)}_{c^{(n-1)}}$.

If $\mathcal{G}$ and $\mathcal{G}_c$ are, respectively, the Lie algebras of G and G_c, let (A, X) denote the general element of $\mathcal{G}_c$, with X in $\mathcal{G}$ and A in the Lie algebra $\mathcal{R}^n$ of Θ.

From (2.3) one then finds that

$$[(A, X), (B, Y)] = (\gamma(X, Y), [X, Y]) \tag{2.4}$$

where $\gamma : \mathcal{G}x\mathcal{G} \to \mathcal{R}^n$ is the infinitesimal of c and satisfies

$$\gamma(X, Y) = -\gamma(Y, X) \tag{2.5a}$$

$$\gamma([X, Y], Z) + \gamma([Y, Z], X) + \gamma([Z, X], Y) = 0. \tag{2.5b}$$

If (X_i), $i = 1, \ldots, \dim G$, is a basis for $\mathcal{G}$ and if (A_α), $\alpha = 1, \ldots, n$, is a basis for $\mathcal{R}^n$ then the nontrivial Lie brackets for $\mathcal{G}_c$ are

$$[X_i, X_j] = X_k C^k_{ij} + A_\alpha d^\alpha_{ij} \tag{2.6}$$

C^k_{ij} being the constants of structure for $\mathcal{G}$.

The n-dimensional Abelian Lie algebra generated by the A_α is the centre of $\mathcal{G}_c$. One has also that

$$\gamma(X_i, X_j) = A_\alpha d^\alpha_{ij} \tag{2.7}$$

and one verifies, from (2.5*a*) and (2.5*b*), that

$$d^\alpha_{ij} = -d^\alpha_{ji} \tag{2.8a}$$

$$d^\alpha_{lk} C^l_{ij} + d^\alpha_{li} C^l_{jk} + d^\alpha_{lj} C^l_{ki} = 0. \tag{2.8b}$$

An infinitesimal $\gamma : \mathcal{G} x \mathcal{G} \longrightarrow \mathcal{R}^n$ is trivial if there exist a real linear form β such that

$$\gamma(X, Y) = \beta([X, Y]). \tag{2.9}$$

Also two infinitesimals which differ by a trivial one are equivalent and give rise to isomorphic extensions.

3. First central extension $G^{(1)}$ of the Galilei group G and acceleration

First of all, let us recall that the general element g of the one spatial dimensional Galilei group is parametrized as follows

$$g = (v, t, x) \tag{3.1}$$

where x, t, v are, respectively, space translation, time translation and Galilean boost from one inertial frame to another, their dimensions being respectively L (for length), T (for time) and LT^{-1}.

The multiplication law is

$$(v', t', x')(v, t, x) = (v' + v, t' + t, x' + v't + x). \tag{3.2}$$

From this we calculate that the Lie algebra $\mathcal{G}$ is generated by the left-invariant vector fields (which, as anyone knows, generate right translations)

$$K = \partial_v \qquad E = \partial_t + v\partial_x \qquad P = \partial_x. \tag{3.3}$$

We then see that the non-trivial Lie bracket is

$$[K, E] = P. \tag{3.4}$$

Using standard methods [8–10] (looking for solutions of the system (2.8*a*)–(2.8*b*)), we can verify that the maximal central extension $\mathcal{G}^{(1)}$ of $\mathcal{G}$ is generated by K, E, P, M, F such that the non-trivial Lie brackets are

$$[K, P] = M \qquad [K, E] = P \qquad [P, E] = F \tag{3.4a}$$

the centre of $\mathcal{G}^{(1)}$ being generated by M and F.

Now, writing the general element $g^{(1)}$ of $G^{(1)}$, the connected Lie group associated to $\mathcal{G}^{(1)}$, as

$$g^{(1)} = e^{\zeta F + \xi M} e^{tE + xP} e^{vK}$$

and using the Baker–Campbell–Hausdorff (BCH) formulae, we find that the multiplication law for $G^{(1)}$ is

$$(v,t,x,\xi,\zeta)(v',t',x',\xi',\zeta') = (v+v', t+t', x+x'+vt', \xi+\xi'+vx' + \tfrac{1}{2}v^2t', \zeta+\zeta'+\tfrac{1}{2}(x-vt)t' - \tfrac{1}{2}tx') \tag{3.2a}$$

where ξ and ζ have respectively L^2T^{-1} and LT as dimensions.

If we define the coboundary $b : G \to \mathcal{R}$ by

$$b(g) = \tfrac{1}{2}tx \tag{3.5}$$

we then verify that the 2-cocycle $c_2 : GxG \to \mathcal{R}$ defined by

$$c_2(g,g') = \frac{t'}{2}(x+vt') + \frac{t}{2}(x'+vt') \tag{3.6}$$

is trivial.

From (3.2*a*) we see that the 2-cocycle $c_1 : GxG \to \mathcal{R}$ is defined by

$$c_1(g,g') = (vx' + \tfrac{1}{2}v^2t', \tfrac{1}{2}t'(x-vt) - \tfrac{1}{2}tx') \tag{3.7}$$

and is equivalent to $c_3 = c_1 + c_2$ where

$$c_3(g,g') = (0, xt' + \tfrac{1}{2}vt'^2). \tag{3.8}$$

The multiplication law (3.2*a*) for $G^{(1)}$ is then equivalent to

$$(v,t,x,\xi,\zeta)(v',t',x',\xi',\zeta') = (v+v', t+t', x+x'+vt', \xi+\xi'+vx' + \tfrac{1}{2}v^2t', \zeta+\zeta'+t'x+\tfrac{1}{2}vt'^2). \tag{3.2b}$$

Starting from the definition of co-adjoint action Ad*:

$$\langle \mathrm{Ad}^*_g(p), \mathrm{Ad}_g(X)\rangle = \langle p, X\rangle; X \in \mathcal{G}, p \in \mathcal{G}^*, g \in G$$

and using the fact that the adjoint action of $G^{(1)}$ on $\mathcal{G}^{(1)}$ is exactly that of G, we verify that the co-adjoint action of G on the central extension's dual Lie algebra $\mathcal{G}^{(1)*}$ is

$$\mathrm{Ad}^*_{(v,t,x)}(k,e,p,m,f) = (k+pt+m(x-vt)+\tfrac{1}{2}ft^2, e-pv+\tfrac{1}{2}mv^2-fx, p-mv+ft, m, f) \tag{3.9}$$

where we have defined the duality $\langle , \rangle : \mathcal{G}^{(1)*}x\mathcal{G}^{(1)} \to \mathcal{R}$ by

$$\langle (k,e,p,m,f), (\delta v, \delta t, \delta x, \delta\xi, \delta\zeta)\rangle = k\delta v + e\delta t + p\delta x + m\delta\xi + f\delta\zeta. \tag{3.10}$$

The right-hand side must have the dimension of an action (mass $\times$(length)^2 $\times$ (time)$^{-1}$). For this reason m, f, p, e and k can be interpreted, respectively, as mass, force, momentum, energy and mass times length.

If we make a change of variables

$$(k, e, p, m, f) \to (q, p, U, m, f)$$

with

$$q = \frac{k}{m} \text{ (position)} \qquad U = e - \frac{1}{2}\frac{p^2}{m} + fq \text{ (potential energy)} \tag{3.11}$$

we then verify from (3.9) that the corresponding G-orbit on $\mathcal{G}^{(1)*}$, which we denote by $O(f, m, U)$, is characterized by three invariants: f, m and U. The orbit is parametrized by the Darboux's coordinates (q, p) such that the symplectic form on $O(f, m, U)$ (see appendix) is

$$\omega = \mathrm{d}p \wedge \mathrm{d}q. \tag{3.12}$$

Also we see from (3.9) that the G-symplectic action on the orbit (i.e. the restriction of the co-adjoint action of G on the orbit) is

$$L_{(v,t,x)}(q, p) = (q + (u - v)t + x + \tfrac{1}{2}\gamma t^2, p - mv + ft) \tag{3.13}$$

where

$$\gamma = \frac{f}{m} \text{(acceleration)} \qquad u = \frac{p}{m} \text{(velocity)} \tag{3.14}$$

which is the fundamental realization of the one spatial dimensional Galilei group on $O(f, m, U)$.

Now, let us study the physics of the orbit. For this we introduce the contact manifold $\mathcal{R} x O(f, m, U)$ endowed with the 2-form

$$\omega(t) = \omega - \mathrm{d}H \wedge \mathrm{d}t \tag{3.15}$$

$H(q, p, t)$ being the Hamiltonian function in the conventional sense. Let $(q(0), p(0), 0)$ denote the state of the system at $t = 0$. Then, by the evolution law

$$(q(t), p(t), t) = L_{(0,t,0)}(q(0), p(0), 0)$$

we find that

$$q(t) = q(0) + ut + \tfrac{1}{2}\gamma t^2 \qquad p(t) = p(0) + ft \tag{3.16}$$

which are Galilei–Newton laws of motion, γ being a *constant acceleration.*

4. Second central extension $\mathcal{G}^{(2)}$ of the Galilei Lie algebra $\mathcal{G}$ and jerk

From (3.4*a*), we verify that *the maximal second central extension* $\mathcal{G}^{(2)}$ of the Galilei Lie algebra is generated by K, E, P, M, F, R, B and S such that the nontrivial Lie brackets are

$$\begin{aligned} &[K,E]=P \qquad [K,P]=M \qquad [K,M]=R \qquad [K,F]=B \\ &[E,P]=-F \qquad [E,M]=-B \qquad [E,F]=-S \end{aligned} \tag{4.1}$$

the centre of $\mathcal{G}^{(2)}$ being generated by B, S and R.

If the general element of $\mathcal{G}^{(2)}$ is written as

$$X=\delta vK+\delta tE+\delta\xi M+\delta\zeta F+\delta rR+\delta bB+\delta sS \tag{4.2}$$

and if we use the fact that

$$\mathrm{Ad}_{\exp tX}=\exp(t\,\mathrm{ad}_X) \tag{4.3}$$

we then verify that the co-adjoint action of $G^{(1)}$ on $\mathcal{G}^{(2)*}$ is given by

$$(k',e',p',f',\rho',\beta',\sigma')=\mathrm{Ad}^*_{(v,t,x,\xi,\zeta)}(k,e,p,m,f,\rho,\beta,\sigma)$$

with

$$k'=k+pt+mx+f\frac{t^2}{2}+\rho\xi+\beta\zeta+\frac{\sigma}{6}t^3 \tag{4.4a}$$

$$e'=e-pv+m\frac{v^2}{2}-f(x+vt)-\rho\frac{v^3}{6}-\beta\left(\xi-\frac{v^2}{2}t\right)-\sigma\left(\zeta+\frac{v}{2}t^2\right) \tag{4.4b}$$

$$p'=p-mv+ft+\rho\frac{v^2}{2}-\beta vt+\sigma\frac{t^2}{2} \tag{4.4c}$$

$$m'=m+\beta t-\rho v \tag{4.4d}$$

$$f'=f-\beta v+\sigma t \tag{4.4e}$$

$$\rho'=\rho \qquad \beta'=\beta \qquad \sigma'=\sigma \tag{4.4f}$$

where the duality between $\mathcal{G}^{(2)}$ and $\mathcal{G}^{(2)*}$ has been defined by

$$\begin{aligned} &\langle(k,e,p,m,f,\rho,\beta,\sigma),(\delta v,\delta t,\delta x,\delta\xi,\delta\zeta,\delta r,\delta b,\delta s)\rangle \\ &\qquad =k\delta v+e\delta t+p\delta x+m\delta\xi+f\delta\zeta+\rho\delta r+\beta\delta b+\sigma\delta s. \end{aligned} \tag{4.5}$$

From the two equations (4.4*d*) and (4.4*e*), we see that ρ, β, σ have, respectively, mass times $(\text{time})^{-1}$, mass times $(\text{velocity})^{-1}$, mass times length times $(\text{time})^{-3}$, and are invariant.

Let us study the physics of the $G^{(1)}$-co-adjoint orbit (see appendix) corresponding to $\rho=\beta=0$. The orbit is characterized by the invariants $j=\sigma/m$ and m. We will denote it by $O(j,m)$. We will see later in the text that j is a jerk.

If we define

$$q=\frac{k}{m} \qquad \gamma=\frac{f}{m} \qquad u=\frac{p}{m} \qquad \phi=\frac{1}{\sigma}\left(e-\frac{p^2}{2m}+f\frac{k}{m}\right) \tag{4.6}$$

we then verify that the orbit is endowed with the symplectic 2-form (in Darboux's coordinates)

$$\omega = \mathrm{d}q \Lambda \mathrm{d}p + \mathrm{d}f \Lambda \mathrm{d}\phi \tag{4.7}$$

and that the $G^{(1)}$-symplectic action on the orbit is

$$L_{(v,t,x,\xi,\zeta)}(q,\phi,p,f) = \left(q + ut + \frac{1}{2}\gamma t^2 + \frac{1}{6}jt^3 + x, \phi + (q+x)t + \frac{u}{2}t^2 + \frac{1}{6}\gamma t^3 \right.$$
$$\left. + \frac{1}{24}jt^4 - \zeta, p - mv + ft + \frac{1}{2}mjt^2, f + mjt\right). \tag{4.8}$$

Now let $(q(0), \phi(0), p(0), f(0), 0)$ be the state of the elementary system at $t = 0$. Then the evolution law

$$(q(t), \phi(t), p(t), f(t), t) = L_{(0,t,0,0,0)}(q(0), \phi(0), p(0), f(0), 0)$$

gives rise to

$$\begin{aligned} q(t) &= q(0) + ut + \tfrac{1}{2}\gamma t^2 + \tfrac{1}{6}jt^3 \\ \phi(t) &= \phi(0) + q(0)t + \tfrac{1}{2}ut^2 + \tfrac{1}{6}\gamma t^3 + \tfrac{1}{24}jt^4 \\ p(t) &= p(0) + f(0)t + \tfrac{1}{2}mjt^2 \\ f(t) &= f(0) + mjt \end{aligned} \tag{4.9}$$

which are the Galilei laws of motion for a nonconstant acceleration γ but with a *constant jerk* j.

We verify that the Hamiltonian equations are

$$\frac{\mathrm{d}p}{\mathrm{d}t} = f \qquad \frac{\mathrm{d}f}{\mathrm{d}t} = mj \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = u + \gamma t + \frac{1}{2}jt^2 \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = q. \tag{4.10}$$

The first three equations are exactly those of Sandin [4]. The fourth one is new and suggests that *the position can be seen as the time rate of change of the observable canonically conjugated with force.*

Appendix. Symplectic form on co-adjoint orbits

We know that the coadjoint $\mathrm{Ad}^* : \mathcal{G}^* x \mathcal{G} \to \mathcal{R}$ of G on $\mathcal{G}^*$ is such that

$$\langle \mathrm{Ad}^*_x(p), Y\rangle = \langle p, [X,, Y]\rangle. \tag{A.1}$$

If

$$p = p_a \varepsilon^a \in \mathcal{G}^* \qquad X = e_a X^a \qquad Y = e_a Y^a \in \mathcal{G}$$

then

$$\langle \mathrm{Ad}^*_x(p), Y\rangle = K_{ab}(p) X^a Y^b \tag{A.2}$$

where

$$K_{ab}(p) = p_d C^d_{ad} \tag{A.3}$$

is the Kirillov 2-form [6] on $\mathcal{G}^*$.

The representation $\rho : \mathcal{G} \to \mathcal{F}(\mathcal{G}^*)$ of $\mathcal{G}$ on the space of vector fields on $\mathcal{G}^*$ defined by

$$\rho(e_a) = K_{ab}(p)\frac{\partial}{\partial p_b} \tag{A.4}$$

is such that

$$\operatorname{Ker} K(p) = \{f \in C^\infty(\mathcal{G}^*, \mathcal{R}) : \rho(X)f = 0, X \in \mathcal{G}\}.$$

This means that $\operatorname{Ker}(K(p))$ is exactly the set of all invariants of $\mathcal{G}$ in $\mathcal{G}^*$. The quotient space $O(p) = \mathcal{G}^*/\operatorname{Ker}(K(p))$, called the co-adjoint orbit of G in $\mathcal{G}^*$, is a symplectic manifold [11, 12]. The symplectic form ω^{ab} is obtained from

$$\omega_{ab}\omega^{bc} = \delta^c_a \tag{A.5}$$

where $\omega_{ab} = K_{ab} \mid O(p)$, i.e. the restriction of the Kirillov form on the orbit.

References

[1] Nzotungicimpaye J 1988 *Lett. Math. Phys.* **15** 101–10
[2] Schot S H 1978 *Am. J. Phys.* **46** 1090–4
[3] Luzader S 1988 *Phys. Teach.* **26** 423, and the following 'Editor's Note'
[4] Sandin T R 1990 *Phys. Teach.* **28** 36–40
[5] Inönü E and Wigner E P 1952 *Il Nuovo Cimento* **IX** 705–18
[6] Levy Leblond J M 1963 *J. Math. Phys.* **4** 776–88
[7] Voisin J 1965 *J. Math. Phys.* **6** 1519
[8] Hamermesh M 1962 *Group Theory and Its Applications to Physical Problems* (New York: Addison-Wesley)
[9] De Stuart J J 1974 *Third Int. Colloquium on Group Theory and Its Applications* Nijmgen p
[10] Ado I D 1947 *Usp. Matematiceskih Nauk.* (N.S.)2 No 6(22) 159–73
[11] Symes W W 1980 *Physica* **1D** 339
[12] Giachetti R 1981 *Rev. Nuovo Cim.* **4** 1–63